TITLE:

Precise construction of fundamental solutions for degenerate operators and their applications (Workshop on the Boltzmann Equation, Microlocal Analysis and Related Topics)

AUTHOR(S):

Iwasaki, Chisato

CITATION:

Iwasaki, Chisato. Precise construction of fundamental solutions for degenerate operators and their applications (Workshop on the Boltzmann Equation, Microlocal Analysis and Related Topics). 数理解析研究所講究録別冊 2017, B67: 155-171

ISSUE DATE:

2017-10

URL:

http://hdl.handle.net/2433/243710

RIGHT:

© 2017 by the Research Institute for Mathematical Sciences, Kyoto University. All rights reserved.

京都大学学術情報リポジトリ KURENAI 紅
Kyoto University Research Information Repository

RIMS Kôkyûroku Bessatsu
B67 (2017), 155–171

Precise construction of fundamental solutions for degenerate operators and their applications

By

Chisato Iwasaki*

§ 1. Introduction

In this note we give the precise form of the fundamental solution for the Fokker-Planck operator and that for degenerate operators of Grushin type according to C.Iwasaki [11] and W.Bauer-K.Furutani-C.Iwasaki [1], [2].

For the Fokker-Planck operator the fundamental solution is constructed as a pseudo-differential operator. We give the precise form of symbol which is useful if one wants to obtain good estimates. In this note we apply the exact form to get the eigenfunction expansion in case the potential is quadratic.

Operators of Grushin type are typical degenerate operators and there are many study on these operators. Our aim is to obtain the exact form of the fundamental solution of operator

$$P_k = -\left(\Delta_x + |x|^{2k}\Delta_z\right).$$

P_k is reduced to some suitable operator $Q_{b,\zeta}$. We obtain the fundamental solution for the heat equation for $Q_{b,\zeta}$, using the modified Bessel function. This part is essential in our study. This method can be applied to construct the inverse of the Kohn-Laplacian.

Both topics are based on the exact form of the fundamental solution of heat equation stated as Theorem 3.2 in this note.

In section 2 we introduce pseudo-differential operators of Weyl symbol. The results on the fundamental solution of degenerate parabolic operators are stated in section 3. Section 4-section 6 are devoted to the Fokker-Planck operator. The results of [1] on operators of Grushin type are stated in section 7. We give a rough sketch of the proof in section 8. In the last section we state the results on the Kohn-Laplacian according to [2].

Received December 30, 2016. Revised April 6, 2017.
2010 Mathematics Subject Classification(s): 35A08, 35P10, 35Q84, 32V05
*University of Hyogo, Himeji 671-2201, Japan.
e-mail: iwasaki@sci.u-hyogo.ac.jp

© 2017 Research Institute for Mathematical Sciences, Kyoto University. All rights reserved.

§ 2. Pseudo-differential operators of Weyl symbol

We use pseudo-differential operators of Weyl symbol in this note.

Definition 2.1. For a symbol $p(x,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^n)(0 \le \delta \le \rho \le 1, \delta < 1)$ we define a pseudo-differential operator $P = p^w(x,D)$ on $\mathbb{R}^n$ is defined as follows:

$$\begin{aligned} Pu(x) &= p^w(x,D)u(x) \\ &= Os - (2\pi)^{-n} \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i(x-y)\cdot\xi} p\left(\frac{x+y}{2},\xi\right) u(y)dyd\xi. \end{aligned}$$

It holds that a product of $p^w(x,D)q^w(x,D)$ is also a pseudo-differential operator of symbol $(p\circ_w q)(x,\xi)$. Moreover we have the following expansion formula:

[Expansion]

If $p \in S^{m_1}_{\rho,\delta}$ $q \in S^{m_2}_{\rho,\delta}$, then for any N we have

$$p\circ_w q = \sum_{j=0}^{N-1} (\frac{1}{2i})^j \sigma_j(p,q) + r_N(p,q),$$

where

$$\sigma_j(p,q) = \sum_{|\alpha|+|\beta|=j} \frac{(-1)^{|\beta|}}{\alpha!\beta!} p^{(\alpha)}_{(\beta)}(x,\xi) q^{(\beta)}_{(\alpha)}(x,\xi) \in S^{m-(\rho-\delta)j}_{\rho,\delta},$$

$r_N(p,q) \in S^{m-(\rho-\delta)N}_{\rho,\delta} \quad (m = m_1 + m_2).$

We note that

$$\sigma_j(p,q) = (-1)^j \sigma_j(q,p) \text{ for any } j$$

and

$$\sigma_1(p,q) = < J\nabla p, \nabla q >, \quad \sigma_2(p,q) = -\frac{1}{2}\mathrm{tr}(JH_pJH_q),$$

where J is a $2n\times 2n$ matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \ \nabla p = {}^t\left(\frac{\partial}{\partial x_1}p, \cdots, \frac{\partial}{\partial x_n}p, \frac{\partial}{\partial \xi_1}p, \cdots, \frac{\partial}{\partial \xi_n}p\right).$$

H_p is the Hesse matrix :

$$H_p = \begin{pmatrix} \partial_x\partial_x p & \partial_x\partial_\xi p \\ \partial_\xi\partial_x p & \partial_\xi\partial_\xi p \end{pmatrix}.$$

§ 3. Fundamental solution for degenerate parabolic operators

A.Melin[13] had shown that

$$\mathrm{Re}(Pu,u) \ge \delta\|u\|^2_{(m-1)/2} - C\|u\|^2_0 \quad (\delta > 0)$$

is equivalent to the following Condition (A) for its symbol $p(x,\xi) \in S^m_{1,0}(\mathbb{R}^n)$.

Condition (A)

$$p_m(x,\xi) \geq 0$$

and there exists a constant $c > 0$ such that

$$\mathrm{Re}\,(p_{m-1}(x,\xi)) + \frac{1}{2}\mathrm{tr}^+(A)(x,\xi) \geq c|\xi|^{m-1} \quad \text{on } \Sigma,$$

where $\Sigma = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n : p_m(x,\xi) = 0\}$, $A(x,\xi) = A = iJH_{p_m}$, $tr^+(A)$ is the sum of positive eigenvalues of A.

Now we have the following theorem for the fundamental solution $e^w(t;x,D)$ for degenerate parabolic operator, which is a pseudo-differenttail operator with parameter t, that is,

$$\frac{d}{dt}e^w(t;x,D) + p^w(x,D)e^w(t;x,D) = 0 \quad in \quad (0,T) \times \mathbb{R}^n,$$

$$e^w(0;x,D) = I.$$

Theorem 3.1. *(C.Iwasaki-N.Iwasaki [9], [10]) Suppose $p(x,\xi) \in S^m_{1,0}(\mathbb{R}^n)$ satisfies condition (A). Then we can construct $e(t;x,\xi) \in S^0_{1/2,1/2}$ such that $S^{-\infty}(t > 0)$ and satisfies the following condition :*
For any positive integer N

$$e(t;x,\xi) - \sum_{j=0}^{N-1} e_j(t;x,\xi) \in S^{-N/2}_{1/2,1/2},$$

$$e_0(t;x,\xi) = e^{\varphi(t;x,\xi)}, \quad e_j(t;x,\xi) \in S^{-j/2}_{1/2,1/2},$$

where

$$\varphi(t;x,\xi) = -p_m t - p_{m-1}t - \frac{1}{2}\mathrm{tr}\left(\log\left(\cosh\left(\frac{At}{2}\right)\right)\right)$$

$$+\frac{it^2}{4} < G(At/2)J\nabla p_m, \nabla p_m >,$$

$$G(x) = (1 - x^{-1}\tanh x)/x.$$

We note that $e_0(t;x,\xi)$ is the symbol of the fundamental solution if $p(x,\xi)$ is a quadratic function.

Theorem 3.2. *If $p(x,\xi)$ is a polynomial with respect $X =^t (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$ such that*

$$p = \frac{1}{2}\langle X, HX\rangle + i\langle X, p_0\rangle + b.$$

Then $E(t) = e^w(t; x, D)$ *is given*

$$e(t;x,\xi) = \frac{e^{-bt}}{\sqrt{\det\cosh(At/2)}} \exp\Big[-i\Big\{\langle J\tanh(At/2)X, X\rangle$$

$$+t\langle J\tanh(At/2)(At/2)^{-1}X, Jp_0\rangle + \frac{t^2}{4}\langle JG(At/2)Jp_0, Jp_0\rangle\Big\}\Big]$$

with $2d \times 2d$ matrix $A = iJH$,

$$J = \begin{pmatrix} 0\, I \\ -I\, 0 \end{pmatrix}.$$

§ 4. The Fokker-Planck operators

We construct the fundamental solution for the Fokker-Planck (Kramers) operator (See : B.Helffer and F.Nier [8] , H.Riskin [14]).

$$\begin{aligned} P &= v\cdot\partial_x - \big(\partial_x V(x)\big)\cdot\partial_v - \triangle_v + \frac{|v|^2}{4} - \frac{n}{2} \\ &= X_0 + \sum_{j=1}^n b_j^* b_j \quad (x, v \in \mathbb{R}^n), \end{aligned}$$

where

$$X_0 = v\cdot\partial_x - \big(\partial_x V(x)\big)\cdot\partial_v, \quad b_j = \partial_{v_j} + \frac{1}{2}v_j.$$

We note that the fundamental solution for the Kolmogorov operator is obtained easily.

Remark. (The Kolmogorov Operator) Let P be an operator as follows:

$$P = \sum_{j=1}^n v_j \frac{\partial}{\partial x_j} - \frac{1}{2}\sum_{j=1}^n \frac{\partial^2}{\partial v_j^2}.$$

Then the fundamental solution $E(t) = e^w(t; x, v, D_x, D_v)$ is a pseudo-differential operator with a symbol

$$e(t;x,v,\xi,\rho) = exp\Big[-\frac{t}{2}|\rho|^2 - iv\cdot\xi t - \frac{t^3}{24}|\xi|^2\Big].$$

§ 5. The symbol of the fundamental solution of the Fokker-Planck operator

The Fokker-Planck operator is a pseudo-differential operator of symbol $p(x, v, \xi, \rho)$ given by

$$\sigma(P) = p(x,v,\xi,\rho) = i\Big(\langle v,\xi\rangle - \langle\partial_x V(x),\rho\rangle\Big) + |\rho|^2 + \frac{|v|^2}{4} - \frac{n}{2}.$$

If $V(x) = a \cdot x + \frac{1}{2}\langle \varepsilon x, x\rangle$ with a symmentric matrix ε, we can apply Theorem 3.2 taking $d = 2n$, $X = ^t(x, v, \xi, \rho)$ and the $4n \times 4n$ matrix A of the form

$$A = \begin{pmatrix} 0 & -I_n & 0 & 0 \\ \varepsilon & 0 & 0 & 2iI_n \\ 0 & 0 & 0 & -\varepsilon \\ 0 & -\frac{i}{2}I_n & I_n & 0 \end{pmatrix}, \quad p_0 = -\begin{pmatrix} 0 \\ 0 \\ 0 \\ a \end{pmatrix}, \quad b = -\frac{n}{2}.$$

Then we have the exact form of the symbol of the fundamental solution:

Theorem 5.1. *(C.Iwasaki [11])*
(I) If $V(x) = a \cdot x$, we have

$$\begin{aligned} & e(t; x, v, \xi, \rho) \\ & = \Big(\frac{1 + e^{-t}}{2}\Big)^{-n} \exp\Big[-2\tanh(t/2)\left(|\rho|^2 + |v/2|^2\right) \\ & \qquad -2\left(t/2 - \tanh(t/2)\right)\left(|\xi|^2 + |a/2|^2\right) \\ & \qquad -2i\tanh(t/2)\left(\langle v, \xi\rangle - \langle a, \rho\rangle\right)\Big]. \end{aligned}$$

(II)If $n = 1$ and $V(x) = \frac{1}{2}\mu x^2 + ax (\mu \neq 0)$, we have

$$e(t; x, v, \xi, \rho) = C(t)\exp\big[-2\phi(t; x, v, \xi, \rho)\big]$$

with $C(t)$ and $\phi(t)$ which are defined as follow depending on the constant μ.

$$\begin{aligned} \phi(t) = {} & F_1(t)\left(|\rho|^2 + |v/2|^2\right) \\ & + F_2(t)\left(|\xi|^2 + |\mu x + a|^2/4\right) \\ & + iF_3(t)\left(\langle v, \xi\rangle - \langle \mu x + a, \rho\rangle\right). \end{aligned}$$

(II-1) If $\mu \neq \frac{1}{4}$, set

$$\lambda_1 = \frac{1}{2}(1 + \delta), \quad \lambda_2 = \frac{1}{2}(1 - \delta). \tag{5.1}$$

Then we have

$$\begin{aligned} C(t) &= 4\Big(1 + e^{-\lambda_1 t}\Big)^{-1}\Big(1 + e^{-\lambda_2 t}\Big)^{-1}, \\ F_1(t) &= \frac{1}{\lambda_1 - \lambda_2}\Big(\lambda_1 g(\lambda_1 t/2) - \lambda_2 g(\lambda_2 t/2)\Big), \\ F_2(t) &= \frac{1}{\lambda_1 - \lambda_2}\Big(\frac{1}{\lambda_2} g(\lambda_2 t/2) - \frac{1}{\lambda_1} g(\lambda_1 t/2)\Big), \\ F_3(t) &= \frac{1}{\lambda_1 - \lambda_2}\Big(g(\lambda_1 t/2) - g(\lambda_2 t/2)\Big) \end{aligned}$$

with $g(x) = \tanh x$*. Here*

$$\delta = \sqrt{1-4\mu} \quad (-\infty < \mu \leq \frac{1}{4}), \qquad \delta = i\sqrt{4\mu - 1} \quad (\mu > \frac{1}{4}).$$

(II-2) If $\mu = \frac{1}{4}$*, we have*

$$\begin{aligned} C(t) &= 4(1+e^{-t/2})^{-2}, \\ F_1(t) &= \tanh(t/4) + \frac{t/4}{\cosh^2(t/4)}, \\ F_2(t) &= 4\left(\tanh(t/4) - \frac{t/4}{\cosh^2(t/4)}\right), \\ F_3(t) &= \frac{t}{2\cosh^2(t/4)}. \end{aligned}$$

We note that the following behavior hold in any case.

$$F_1(t) = \frac{1}{2}t + O(t^3), \quad F_2(t) = \frac{1}{24}t^3 + O(t^5),$$
$$F_1(t) \geq 0, \quad F_2(t) \geq 0.$$

§ 6. The eigenvalues of the operator P

In this section we show the eigenfunction expansion of Fokker-Planck operator by the assertion Theorem 5.1 and the following key lemma.

Lemma 6.1. *(the key Lemma) If the symbol of a pseudo-differential of the form* $e(x,\xi) = h(x,\xi)g(x,\xi)$*, then the kernel of the pseudo-differential operator* $E = e^w(x,D)$ *is given by*

$$\begin{aligned} &(2\pi)^{-d}\int_{\mathbf{R^d}} e^{i(x-x')\cdot\xi} e\left(\frac{x+x'}{2}, \xi\right) d\xi \\ &= h\left(\frac{r}{2}, -i\partial_q\right)\tilde{g}\left(\frac{r}{2}, q\right)|_{r=x+x', q=x-x'}, \end{aligned}$$

where

$$\tilde{g}\left(\frac{r}{2}, q\right) = (2\pi)^{-d}\int_{\mathbf{R^d}} e^{iq\cdot\xi} g\left(\frac{r}{2}, \xi\right) d\xi.$$

In our case we choose $g = g(x, v, \xi, \rho)$ in Lemma 6.1 as follows:
(I)If $\mu > 0$, we take

$$g = \lim_{t\to\infty} e(t; x, v, \xi, \rho).$$

Then

$$\tilde{g} = \psi(x, v)\psi(x', v'),$$

where

$$\psi(x, v) = (\frac{\sqrt{\mu}}{2\pi})^{1/2} \exp\Big[-\frac{1}{4}\{v^2 + \mu(x + \frac{a}{\mu})^2\}\Big].$$

(II)If $\mu < 0$, we take

$$g = \lim_{t\to\infty}\{e^{|\lambda_2|t} e(t; x, v, \xi, \rho)\}.$$

Then

$$\tilde{g} = \psi(x, v)\tilde{\psi}(x', v'),$$

where

$$\psi(x, v) = \left(\frac{\delta\sqrt{-\mu}}{2\pi}\right)^{1/2} \exp\Big\{-\frac{\delta}{4}v^2 + \frac{\delta}{4\mu}(\mu x + a)^2 - v(\mu x + a)\Big\},$$

$$\tilde{\psi}(x, v) = \left(\frac{\delta\sqrt{-\mu}}{2\pi}\right)^{1/2} \exp\Big\{-\frac{\delta}{4}v^2 + \frac{\delta}{4\mu}(\mu x + a)^2 + v(\mu x + a)\Big\}$$

with

$$\delta = \sqrt{1 - 4\mu}.$$

We obtain the following expansions of the kernel $K(t; (x, v), (x', v'))$ of the fundamental solution corresponding to the constant μ.

§ 6.1. In case $\mu = \frac{1}{4}$

Definition 6.2.

$$a_x = 2\partial_x + \frac{x}{4} + a, \quad b_v = \partial_v + \frac{v}{2},$$

$$M = \frac{1}{\sqrt{2}}(b_v - a_x), \quad N = \frac{1}{\sqrt{2}}(b_v + a_x)$$

$$\psi_{m,k}(x, v) = \frac{(M^*)^m}{\sqrt{m!}} \frac{(N^*)^k}{\sqrt{k!}} \psi(x, v), \quad m \geq 0,\ k \geq 0,$$

$$\varphi_{n,k}(x, v) = \psi_{n-k,k}(x, v), \quad 0 \leq k \leq n.$$

By the statement of the following theorem $\Big\{\varphi_{n,k}\Big\}_{k=0}^{n}$ are the set of orthogonal generalized eigenfunctions whose eigenvalue is $n/2$.

Theorem 6.3. *(1)*

$$P\varphi_{n,k}(x, v) = \frac{n}{2}\varphi_{n,k}(x, v) + \sqrt{k(n - k + 1)}\varphi_{n,k-1}(x, v)$$

(2)

$$\langle \varphi_{m,k}, \varphi_{n,\ell} \rangle = \delta_{m,n}\delta_{k,\ell}$$

(3)

$$K(t;(x,v),(x',v'))$$
$$= \sum_{n=0}^{\infty} e^{-nt/2}\Big(\sum_{k=0}^{n} \frac{(-t)^k}{k!} \sum_{\ell=0}^{n-k} \sqrt{\frac{(n-\ell)!(\ell+k)!}{\ell!(n-\ell-k)!}}$$
$$\varphi_{n,\ell}(x,v)\varphi_{n,\ell+k}(x',v')\Big)$$

§ 6.2. In case $\mu > 0$, $\mu \neq \frac{1}{4}$

Let λ_1 and λ_2 are constants defined in (5.1).

Definition 6.4.

$$a_x = \partial_x + \frac{\mu x + a}{2}, \quad b_v = \partial_v + \frac{v}{2}, \tag{6.1}$$

$$M_\lambda = \frac{1}{\sqrt{\delta}}(\sqrt{\lambda} b_v - \frac{1}{\sqrt{\lambda}} a_x), \quad N_\lambda = \frac{1}{\sqrt{\delta}}(\sqrt{\lambda} b_v + \frac{1}{\sqrt{\lambda}} a_x), \tag{6.2}$$

$$\psi_{n,k}(x,v) = \frac{((M_{\lambda_1})^*)^n}{\sqrt{n!}} \frac{((-M_{\lambda_2})^*)^k}{\sqrt{k!}} \psi(x,v),$$
$$\psi^{n,k}(x,v) = \frac{((N_{\lambda_1})^*)^n}{\sqrt{n!}} \frac{((N_{\lambda_2})^*)^k}{\sqrt{k!}} \psi(x,v).$$

By the statement of the following theorem $\Big\{\psi_{n,k}\Big\}$ are the set of eigenfunctions whose eigenvalue is $\lambda_1 n + \lambda_2 k$.

Theorem 6.5. *(1)*

$$P\psi_{n,k}(x,v) = (\lambda_1 n + \lambda_2 k)\psi_{n,k}(x,v)$$

(2)

$$\langle \psi_{m,k}, \psi^{n,\ell} \rangle = \delta_{m,n}\delta_{k,\ell}$$

(3)

$$K(t;(x,v),(x',v'))$$
$$= \sum_{n_1,n_2=0}^{\infty} e^{-(\lambda_1 n_1 + \lambda_2 n_2)t} \psi_{n_1,n_2}(x,v)\psi^{n_1,n_2}(x',v')$$

§ 6.3. In case $\mu < 0$

Let λ_1 and λ_2 are constants defined in (5.1). a_x, b_v, M_λ and N_λ are operators defined as (6.1) and (6.2).

Definition 6.6.

$$\psi_{n,k}(x,v) = \frac{((M_{\lambda_1})^*)^n}{\sqrt{n!}} \frac{(-M_{|\lambda_2|})^k}{\sqrt{k!}} \psi(x,v),$$

$$\psi^{n,k}(x,v) = \frac{((N_{\lambda_1})^*)^n}{\sqrt{n!}} \frac{(N_{|\lambda_2|})^k}{\sqrt{k!}} \tilde{\psi}(x,v).$$

By the statement of the following theorem $\left\{\psi_{n,k}\right\}$ are the set of eigenfunctions whose eigenvalue is $\lambda_1 n + |\lambda_2|(k+1)$.

Theorem 6.7. *(1)*

$$P\psi_{n,k}(x,v) = (\lambda_1 n + |\lambda_2|(k+1))\, \psi_{n,k}(x,v)$$

(2)

$$\langle \psi_{m,k}, \psi^{n,\ell} \rangle = \delta_{m,n}\delta_{k,\ell}$$

(3)

$$K(t;(x,v),(x',v')) = \sum_{n_1,n_2=0}^{\infty} e^{-(\lambda_1 n_1 + |\lambda_2|(n_2+1))t} \psi_{n_1,n_2}(x,v)\psi^{n_1,n_2}(x',v')$$

§ 7. Operators of Grushin type and the precise form of their inverse

We will show a way of construction of the fundamental solution to the following degenerate operator: This is the typical model which is degenerate of order $2k (k \in \mathbb{N})$:

$$P_k = -\left(\Delta_x + |x|^{2k}\Delta_z\right)$$

$$x \in \mathbb{R}^N \quad (N \geq 2), \quad z \in \mathbb{R}^\ell$$

M.I. Visik and V.V. Grushin [15] studied operators of polynomial coefficients which are degenerate on submanifold. They researched hypoellipticity, local solvability and estimates of operators of this form.

If $k = 1$, we can apply Theorem 3.2 as follows:
Choosing $X =^t (x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N$, $p_0 = b = 0$ and $(2N) \times (2N)$ matrix

$$H = 2\begin{pmatrix} |\zeta|^2 & 0 \\ 0 & I \end{pmatrix},$$

we have the symbol of the fundamental solution $e_\zeta(t;x,\xi)$ with parameter $\zeta\in\mathbb{R}^\ell$ as

$$\begin{aligned}e_\zeta(t;x,\xi) &= \frac{1}{\sqrt{\det\cosh(At/2)}}\exp\left(-i\langle J\tanh(At/2)X,X\rangle\right)\\ &= \left(\frac{1}{\cosh(|\zeta|t)}\right)^N \exp\left\{-\frac{\tanh(|\zeta|t)}{|\zeta|}(|x|^2|\zeta|^2+|\xi|^2)\right\}.\end{aligned}$$

The kernel $K_\zeta(t,x,x')$ of pseudo-differential operator $e^w_\zeta(t;x,D)$ with symbol $e_\zeta(t;x,\xi)$ is given by

$$\begin{aligned}K_\zeta(t,x,x') = \frac{|\zeta|}{2\pi\sinh(2|\zeta|t)}\exp\Big\{ &-|\zeta|\coth(2|\zeta|t)\frac{|x|^2+|x'|^2}{2}\\ &+\frac{|\zeta|}{\sinh(2|\zeta|t)} < x,x' >\Big\}.\end{aligned}$$

Then we have

$$e^{-tP_1}\phi(x,z) = \frac{1}{(2\pi)^\ell}\int_{\mathbb{R}^{2\ell+N}} e^{i(z-z')\cdot\zeta}K_\zeta(t,x,x')\phi(x',z')d\zeta dx'dz'.$$

We obtain the exact form for of the inverse of P_k.

Theorem 7.1. *(W.Bauer-K.Furutani-C.Iwasaki [1]) Suppose $(x,z)\neq(x',z')\in\mathbb{R}^N\times\mathbb{R}^\ell$.*

(I) If $\ell=2q$, then

$$(P_k)^{-1}((x,z),(x',z')) = \frac{k+1}{2\pi}F_{q,k,N}(\beta,\gamma).$$

(II)If $\ell=2q-1$, then

$$(P_k)^{-1}((x,z),(x',z')) = \frac{\sqrt{2}}{2\pi}\int_\beta^\infty F_{q,k,N}(u,\gamma)\frac{1}{\sqrt{u-\beta}}du.$$

Here

$$\begin{aligned}\gamma =& |x|^{k+1}|x'|^{k+1},\\ \beta =& \frac{1}{2}\left\{|x|^{2(k+1)}+|x'|^{2(k+1)}+(k+1)^2|z-z'|^2\right\},\end{aligned}$$

$$\begin{aligned}F_{q,k,N}(u,\gamma) &= \left(\frac{-(k+1)^2}{2\pi}\frac{\partial}{\partial u}\right)^{q-1}F_{1,k,N}(u,\gamma) \quad (q\geq 2),\\ F_{1,k,N}(u,\gamma) &= \frac{1}{|S^{N-1}|}\frac{u_+-u_-}{(u_+^{k+1}-u_-^{k+1})\left\{u_++u_--2<x,x'>\right\}^{N/2}},\end{aligned}$$

where

$$u_\pm = u_\pm(u,\gamma) = (u \pm \sqrt{u^2-\gamma^2})^{1/(k+1)}$$

and $|S^n|$ *is the volume of* n*-dimensional sphere.*

Remark. In case $\ell = 2$

$$(P_k)^{-1}((x,z),(x',z')) = \frac{k+1}{2\pi} F_{1,k,N}(\beta,\gamma).$$

$$F_{1,0,N}(\beta,\gamma) = \frac{1}{|(x,z)-(x',z')|^N}, \quad F_{1,1,N}(\beta,\gamma) = \frac{1}{\rho(\rho - 2 < x, x' >)^{N/2}},$$

where

$$\rho = \sqrt{(|x|^2+|x'|^2)^2 + 4|z-z'|^2}.$$

The singularity of operator $F_{1,k,N}(\beta,\gamma)$ near the diagonal set is as follows:

Remark. (1)If $x = x' = 0$, then $F_{1,k,N}(\beta,\gamma) \sim |z-z'|^{-\frac{2k+N}{k+1}}$.
(2)If $z - z' = 0$, then we have

$$F_{1,k,N}(\beta,\gamma) \sim \begin{cases} |x-x'|^{-N} & (x \neq 0, x' \neq 0), \\ |x|^{-(N+2k)} & (x' = 0). \end{cases}$$

We can show that $(P_k)^{-1}((x,z),(x',z'))$ is local L^1 and P_k is an essentially self adjoint operator.

§ 8. Sketch of proof

In this note we restrict ourselves to $N = 2$. See [1] for general case.

If we choose the polar coordinate $x = (r,\theta)$, then we have

$$P_k = -\left(\Delta_x + |x|^{2k}\Delta_z\right) = -\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + r^{2k}\Delta_z\right)$$

for a nonnegative integer k.

Proposition 8.1. *Set* $\tilde r = r^b$, $b = k+1$. *Then we have*

$$\begin{aligned} P_k =& -b^2\tilde r^{2-2/b}\left(\frac{\partial^2}{\partial \tilde r^2} + \frac{1}{\tilde r}\frac{\partial}{\partial \tilde r} + \frac{1}{b^2\tilde r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{b^2}\Delta_z\right) \\ =& b^2\tilde r^{2-2/b} Q_b. \end{aligned}$$

Then $(P_k)^{-1}$ is given by

$$((P_k)^{-1}f)(r) = (Q_b)^{-1} \circ \frac{1}{b^2\tilde{r}^{2-2/b}} \circ F,$$

where

$$F(\tilde{r}, \theta, z) = f(r, \theta, z).$$

Set operator $Q_{b,\zeta}$ with parameter ζ as follows :

$$Q_{b,\zeta} = -\left(\frac{\partial^2}{\partial\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}} + \frac{1}{b^2\tilde{r}^2}\frac{\partial^2}{\partial\theta^2} - \frac{1}{b^2}|\zeta|^2\right)$$

$$(\tilde{r} > 0, 0 \leq \theta < 2\pi).$$

Then

$$(Q_b)^{-1}F(\tilde{r}, z) = \frac{1}{2\pi}\int_{\mathbb{R}^\ell} e^{i(z-z')\cdot\zeta}((Q_{b,\zeta})^{-1}F)(\tilde{r}, \theta, z')dz'd\zeta.$$

We reduce our problem to construction of $(Q_{b,\zeta})^{-1}$.

Now consider to obtain the fundamental solution of the initial problem:

$$\left(\frac{\partial}{\partial t} + Q_{b,\zeta}\right)U^\zeta(t) = 0 \quad (t > 0, \tilde{r} > 0, 0 \leq \theta < 2\pi),$$

$$U^\zeta(0) = I.$$

Set

$$Q_{b,\zeta}(n) = -\left(\frac{\partial^2}{\partial\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}} - \frac{n^2}{b^2}\frac{1}{\tilde{r}^2} - \frac{1}{b^2}|\zeta|^2\right) \quad (n \in \mathbb{Z}).$$

We will construct the fundamental solution $U_n^\zeta(t)$ for $Q_{b,\zeta}(n)$:

$$\left(\frac{\partial}{\partial t} + Q_{b,\zeta}(n)\right)U_n^\zeta(t) = 0 \quad (t > 0, \tilde{r} > 0),$$

$$U_n^\zeta(0) = I.$$

The following proposition is one of the key proposition.

Proposition 8.2. *$U_n^\zeta(t)$ has the kernel function $U_n^\zeta(t, \tilde{r}, \tilde{R})$*

$$U_n^\zeta(t, \tilde{r}, \tilde{R}) = \frac{1}{2t}\exp\left\{-\frac{1}{4t}(\tilde{r}^2 + \tilde{R}^2) - \frac{t|\zeta|^2}{b^2}\right\} I_{|n|/b}\left(\frac{\tilde{r}\tilde{R}}{2t}\right),$$

$$(b = k+1)$$

where $I_\mu(z)$ is the modified Bessel function of order μ.

For the proof of Proposition 8.2 we use the fact $J_\nu(\sqrt{\lambda}r)$ is the eigen function with eigenvalue λ:

$$-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{\nu^2}{r^2}\right)J_\nu(\sqrt{\lambda}r)=\lambda J_\nu(\sqrt{\lambda}r)$$

and Weber's second exponential integral (See [16]), that is

$$\int_0^\infty \exp(-\lambda t)J_\nu(\sqrt{\lambda}r)J_\nu(\sqrt{\lambda}R)d\lambda = \frac{1}{t}\exp\left(-\frac{r^2+R^2}{4t}\right)I_\nu\left(\frac{rR}{2t}\right).$$

By Proposition 8.2 the inverse of $Q_{b,\zeta}$ is obtained as

$$(Q_{b,\zeta})^{-1}=\int_0^\infty U^\zeta(t)dt=\frac{1}{2\pi}\int_0^\infty \sum_{n\in\mathbb{Z}}U_n^\zeta(t,\tilde{r},\tilde{R})dte^{in(\theta-\omega)}.$$

Finally we have the following expression:

$(Q_b)^{-1}$ is an integral operator with kernel

$$(Q_b)^{-1}((\tilde{r},\theta,z),(\tilde{R},\omega,z'))= = \left(\frac{1}{2\pi}\right)^{\ell+1}\int_{\mathbb{R}^\ell}\int_0^\infty e^{i(z-z')\cdot\zeta}\sum_{n\in\mathbb{Z}}U_n^\zeta(t,\tilde{r},\tilde{R})dtd\zeta e^{in(\theta-\omega)}.$$

We have the assertion of Theorem 7.1 by calculation of the right hand side of the above equation using the following formula of the modified Bessel function.

$$I_\mu(z)=\left(\frac{1}{2\pi i}\right)\int_{\infty-i\pi}^{\infty+i\pi}e^{z\cosh w-\mu w}dw.$$

§ 9. Application to the Kohn-Laplacian model

Let M be a complex manifold with boundary. Consider Dolbeault complex on M with $\overline{\partial}$-Neumann problem. It is well-known that $\overline{\partial}$-Neumann problem is not a coercive boundary value problem. This problem can be reduced to problem on a degenerate operator on the boudary ∂M(J.J.Kohn and H.Rossi [12]).

The Kohn-Laplacian is the following form:

$$Z=\frac{\partial}{\partial z}+im\bar{z}|z|^{2m-2}\frac{\partial}{\partial s},\quad \bar{Z}=\frac{\partial}{\partial \bar{z}}-imz|z|^{2m-2}\frac{\partial}{\partial s},$$

$$\begin{aligned}\Delta_\lambda &= -\frac{1}{2}(Z\bar{Z}+\bar{Z}Z) - \frac{1}{2}\lambda[Z,\bar{Z}] \\ &= -\frac{1}{4}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) + mr^{2m-2}\frac{\partial^2}{\partial s\partial\theta} \\ &\quad -m^2 r^{4m-2}\frac{\partial^2}{\partial s^2} + im^2\lambda r^{2m-2}\frac{\partial}{\partial s},\end{aligned}$$

where $m \in \mathbb{N}, \quad \lambda \in \mathbb{C}(-1 < \Re\lambda < 1)$.

The case $m = 1$ corresponds M is strongly pseudoconvex (the Levi form is positive definite).

If we set

$$z = r\cos\theta + ir\sin\theta, \quad \tilde{r} = r^m, \text{ then } \quad \Delta_\lambda = m^2\tilde{r}^{2-2/m}Q,$$

where

$$Q = -\frac{1}{4}\left(\frac{\partial^2}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}} + \frac{1}{m^2\tilde{r}^2}\frac{\partial^2}{\partial \theta^2}\right) + \frac{1}{m}\frac{\partial^2}{\partial s\partial\theta} - \tilde{r}^2\frac{\partial^2}{\partial s^2} + i\lambda\frac{\partial}{\partial s}.$$

Using the similar argument to the previous section, we have the inverse $(Q)^{-1}$ constructed an integral operator with kernel

$$\begin{aligned}&(Q)^{-1}((\tilde{r},\theta,s),(\tilde{R},\omega,s')) \\ &\quad = (\frac{1}{2\pi})^2\int_{\mathbb{R}}\int_0^\infty e^{i(s-s')\sigma}\sum_{n\in\mathbb{Z}} U_n^\sigma(t,\tilde{r},\tilde{R})dtd\sigma e^{in(\theta-\omega)},\end{aligned}$$

where $U_n^\sigma(t,\tilde{r},\tilde{R})$ is the kernel of the fundamental solution $U_n^\sigma(t)$ of the following equation:

$$\begin{aligned}\left(\frac{\partial}{\partial t} + Q_\sigma(n)\right)U_n^\sigma(t) = 0 \quad (\tilde{r} > 0, t > 0), \\ U_n^\sigma(0) = I.\end{aligned}$$

Here

$$Q_\sigma(n) = \frac{1}{4}\left(\frac{\partial^2}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}} - \frac{n^2}{m^2}\frac{1}{\tilde{r}^2}\right) + \tilde{r}^2\sigma^2 - (\lambda + \frac{n}{m})\sigma.$$

We have the following assertion for $U_n^\sigma(t)$.

Proposition 9.1. *$U_n^\sigma(t)$ has the kernel function $U_n^\sigma(t,\tilde{r},\tilde{R})$*

$$\begin{aligned}U_n^\sigma(t,\tilde{r},\tilde{R}) &= \frac{2\sigma}{\sinh(\sigma t)} I_{|n|/m}\left(\frac{2\sigma\tilde{r}\tilde{R}}{\sinh(\sigma t)}\right) \\ &\quad \times \exp\left\{-\sigma\coth(\sigma t)(\tilde{r}^2+\tilde{R}^2) + (\lambda+\frac{n}{m})\sigma t\right\}.\end{aligned}$$

The following argument is a hint of the proof of Proposition 9.1. If $k = 1$, the kernel $K_\zeta(t, x, x')$ is of the form

$$\begin{aligned}K_\zeta(t,x,x') =& \frac{|\zeta|}{2\pi\sinh(2|\zeta|t)}\exp\Big\{-|\zeta|\coth(2|\zeta|t)\frac{|x|^2+|x'|^2}{2}\\ &+\frac{|\zeta|}{\sinh(2|\zeta|t)} < x,x' > \Big\}\\ =& \frac{\zeta}{2\pi\sinh(2|\zeta|t)}\exp\Big\{-|\zeta|\coth(2|\zeta|t)\frac{r^2+R^2}{2}\\ &+\frac{|\zeta|rR}{\sinh(2|\zeta|t)}\cos(\theta-\omega)\Big\}\\ &= K_\zeta(r,\theta;R,\omega)\end{aligned}$$

for $x = (r, \theta), x' = (R, \omega)$. We can show for $n \in \mathbb{Z}$

$$\begin{aligned}\int_0^{2\pi} & K_\zeta(r,\theta;R,\omega)e^{in\omega}d\omega\\ &= \frac{|\zeta|}{\sinh(2|\zeta|t)}\exp\left\{-|\zeta|\coth(2|\zeta|t)\frac{r^2+R^2}{2}\right\}\\ &\times I_n\left(\frac{|\zeta|rR}{\sinh(2|\zeta|t)}\right)e^{in\theta}.\end{aligned}$$

By the similar method in the previous section we have

Theorem 9.2. *(W.Bauer-K.Furutani-C.Iwasaki [2])*

$$\Delta_\lambda^{-1}((z,s),(z',s')) = \frac{(A/|A|)^\lambda}{2m\pi^2|A|}I(\lambda;(z,s),(z',s')),$$

with

$$I(\lambda;(z,s),(z',s')) = \int_0^\infty t^{\frac{m(\lambda+1)}{2}-1}\cdot F(t,P,\overline{P};\gamma)dt.$$

Here

$$A = \frac{1}{2}\Big\{(|z|^{2m}+|z'|^{2m})+i(s-s')\Big\},\ \gamma = \frac{|z|^m|z'|^m}{|A|},\ P = \frac{z\bar{z'}}{\bar{A}^{1/m}}$$

and

$$F(t) = F(t,P,\overline{P};\gamma) := \frac{g_+(t)-g_-(t)}{g_+^m(t)-g_-^m(t)}\cdot\frac{m/2}{g_+(t)+g_-(t)-(tP+\overline{P})},$$

where

$$g_\pm(t) = g_\pm(t;\gamma) := \left\{\frac{1+t^m}{2}\pm\sqrt{\frac{(1+t^m)^2}{4}-\gamma^2t^m}\right\}^{\frac{1}{m}}.$$

Corollary 9.3. *(I) If $z=0$ or $z'=0$, then*

$$\Delta_\lambda^{-1}\big((z,s),(z',s')\big) = \frac{1}{4m\pi^2|A|}\frac{\pi}{\cos(\lambda\pi/2)}\left(\frac{A}{|A|}\right)^{\lambda}.$$

(II) If $m=1$, then

$$\Delta_\lambda^{-1}\big((z,s),(z',s')\big) = \frac{1}{4\pi^2|P_1|}\frac{\pi}{\cos(\lambda\pi/2)}\left(\frac{P_1}{|P_1|}\right)^{\lambda}.$$

(III) (cf. P. Greiner [7]) If $m=2$ and $\lambda=0$, then

$$\Delta_0^{-1}\big((z,s),(z',s')\big) = \begin{cases} \frac{1}{4\pi^2|P_2|}\left\{\arccos\Big(-\frac{A^{1/2}z\bar{z'}+\bar{A}^{1/2}\bar{z}z')}{|z|^2|z'|^2+|A|}\Big)\right\}, & \text{if } (z',s')\neq(-z,s) \\ \frac{1}{8\pi^2}\frac{1}{|A|}, & \text{if } (z',s')=(-z,s), \end{cases}$$

$$A := \frac{1}{2}\left\{(|z|^{2m}+|z'|^{2m})+i(s-s')\right\} \quad (m\in\mathbb{N}),$$
$$P_m := \frac{1}{2}\left\{|z^m-z'^m|^2+i\big(s-s'+2\mathrm{Im}\ (z^m\bar{z}'^m)\big)\right\} \quad (m\in\mathbb{N}).$$

By the fact that $F(t)$ satisfies

$$F^{(\ell)}(t) = \begin{cases} O(t^{-m-\ell}) & \text{as } t\to\infty, \\ O(1) & \text{as } t\to 0, \end{cases}$$

we get

Theorem 9.4. *Δ_λ^{-1} admits a meromorphic extension from $\mathrm{Re}\ (\lambda)\in(-1,1)$ to the complex plane with at most simple poles in the set $\mathcal{P}$ defined in*

$$\mathcal{P} := \left\{\pm\left(1+\frac{2j}{m}\right)\ :\ j=0,1,2,\cdots\right\}.$$

The corresponding residues with $j=0,1,2,\cdots$ are given by

$$\mathrm{Res}\ \left\{\Delta_\lambda^{-1},\lambda=-1-\frac{2j}{m}\right\} = \frac{(A/|A|)^{-1-2j/m}}{m^2\pi^2 j!|A|}\cdot F^{(j)}(0;P,\overline{P}),$$
$$\mathrm{Res}\ \left\{\Delta_\lambda^{-1},\lambda=1+\frac{2j}{m}\right\} = -\frac{(A/|A|)^{1+2j/m}}{m^2\pi^2 j!|A|}\cdot F^{(j)}(0;\overline{P},P).$$

In particular,

$$\mathrm{Res}\ \big\{\Delta_\lambda^{-1},\lambda=-1\big\} = \frac{1}{2m\pi^2 A}\cdot\frac{1}{1-\overline{P}},$$
$$\mathrm{Res}\ \big\{\Delta_\lambda^{-1},\lambda=1\big\} = -\frac{1}{2m\pi^2 \bar{A}}\cdot\frac{1}{1-P}.$$

References

[1] W.Bauer-K.Furutani-C.Iwasaki: *The fundamental solution of a higher step Grushin type operator*, Advances in Math.271(2015), 188-234.

[2] W.Bauer-K.Furutani-C.Iwasaki: *The inverse of a parameter family of degenerate operators and applications to the Kohn-Laplacian*, Advances in Math., 277(2015), 283-337.

[3] G. B. Folland and E. M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974), 429–522.

[4] K.Furutani-C.Iwasaki-T.Kagawa: *An action function for a higher step Grushin operator*, J. Geomery and Physics 62(2012), 1949-1976.

[5] V.V.Grushin: *On a class of hypoelliptic operators*, Mat. Sb.,83(125) (1970), 458-476, Math. USSR Sb.,12(1970), 456-473.

[6] V.V.Grushin: *On a class of elliptic pseudodifferential operators degenerate on a submanifold*, Mat. Sb.,84(126)(1971), 163-195, Math. USSR Sb.,13(1971), 155-185.

[7] P. C. Greiner , *A fundamental solution for a non-elliptic partial differential operator*, Can.J.math. 31(1979), 1107-1120.

[8] B.Helffer and F.Nier: *Hypoellipticity and spectral theory for Fokker-Plancsk operators and Witten Laplacians* , Lecture Notes in Math. 1862 (2005), Springer.

[9] C.Iwasaki and N.Iwasaki. *Parametrix for a degenerate parabolic equation*, Proc.Japan Acad. **55** (1979), 237-240.

[10] C.Iwasaki and N.Iwasaki. *Parametrix for a Degenerate Parabolic Equation and its Application to the Asymptotic Behavior of Spectral Functions for Stationary Problems*, Publ.Res.Inst.Math.Sci.**17** (1981), 557-655.

[11] C.Iwasaki: *A Representaion of the fundamental solution for the Fokker-Planck Equation and Its application*, Fourier Analysis, Trends in Mathematics, (2014), 211-233.

[12] J.J.Kohn and H.Rossi, *On the extension of holomorphic functions from the boundary of a complex manifold* , Ann. of Math. 81(1965), 451-472.

[13] A.Melin. *Lower bounds for pseudo-differential operators*, Ark.Mat. **9** (1971), 117-140.

[14] H.Risken, *The Fokker-Planck equation Methods of solution and applications,* The second edition (1989), Springer.

[15] M.I. Visik and V.V. Grushin, *On a class of degenerate elliptic equations*, Mat. Sb. vol. 79.,(121)(1969), 3-36, Math. USSR. Sb. 8 (1969), 1-32.

[16] G.N.Watson: *A Treatise on the Theory of Bessel functions*, The second edition (1944), Cambridge University Press.